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ON TOTALLY POSITIVE UNITS OF REAL HOLOMORPHY RINGS

BY

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ABSTRACT

In this paper we will show that every totally positive unit of the real holomorphy ring of a formally real field is a sum of $2n$-th powers of totally positive units for all natural numbers n. Moreover, in the case $n = 1$ we give a bound on the number of summands required in such a representation.

Introduction

At the Oberwolfach conference on "real algebraic geometry" in 1987 H.-W. Schülting raised the following question:

PROBLEM: *Let $f, g \in \mathbb{R}[X]$ be of the same degree and without real zeroes. Assume that f/g is positive definite. Are there $f_i, g_i \in \mathbb{R}[X]$ $(1 \le i \le n)$ without real zeroes with $\deg f_i = \deg g_i$ $(1 \le i \le n)$ such that*

$$\frac{f}{g} = \sum_{i=1}^{n} \left(\frac{f_i}{g_i}\right)^2$$

for a certain natural number n?

Since a rational function $h \in \mathbb{R}(X)$ is a totally positive unit of the real holomorphy ring $H(\mathbb{R}(X))$ of $\mathbb{R}(X)$ iff it is positive definite and of the form $h = f/g$ for some $f, g \in \mathbb{R}[X]$ of the same degree and without real zeroes we can generalize the above problem:

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PROBLEM: *Is every totally positive unit of the real holomorphy ring of a formally real field a sum of squares of units ?*

The aim of this paper is to give a positive answer to both problems. We actually prove the stronger result that every totally positive unit of the real holomorphy ring is a sum of $2n$-th powers of totally positive units for arbitrary $n \in \mathbb{N}$.

1. Basic notations and preliminary remarks

Throughout this paper K will be a formally real field. Thus $\mathrm{char}(K) = 0$ and we may assume that $\mathbb{Q} \subset K$. The real holomorphy ring of K will be denoted by $H(K)$. By definition $H(K)$ is the intersection of all residually real valuation rings (i.e. valuation rings with formally real residue class field) of K. Various descriptions of $H(K)$ are well-known, e.g.

$$H(K) = \{a \in K \mid\ n \pm a \in \sum K^2 \text{ for some } n \in \mathbb{N}\} \tag{1}$$

(see e.g. Theorem (2.16),[B1]). Here $\sum K^2$ is the set of sums of squares of elements of K. It is also the set of totally positive elements of K (i.e. elements which are positive w.r.t. every ordering of K). See e.g. [P], Corollary 1.9. Obviously $\mathbb{Q} \subset H(K)$.

For $a, b \in K$ we define

$$a \leq b \Longleftrightarrow b - a \in \sum K^2.$$

Clearly $\leq$ is a preordering of K. It is easy to see that

$$H(K)^\times \cap \sum K^2 = \{a \in K \mid\ \frac{1}{n} \leq a \leq n \quad \text{for some } n \in \mathbb{N}\}. \tag{2}$$

This is the set of the totally positive units of $H(K)$. We denote it by $\mathrm{U}^+(K)$. The following lemma is an immediate consequence of (1) and (2).

LEMMA 1.1:

(i) *For $a_1, \ldots, a_n \in K$ we have $\sum_{i=1}^{n} a_i^2 \in H(K) \Leftrightarrow a_1, \ldots, a_n \in H(K)$.*

(ii) *For $a \in \mathrm{U}^+(K)$ and $b \in H(K)$ such that $a \leq b$ also $b \in \mathrm{U}^+(K)$.*

2. Representation as sums of squares

In this section we show that every totally positive unit of $H(K)$ is a sum of squares of totally positive units of $H(K)$. The crucial step in the whole paper is the following lemma. The proof of this lemma is similar to a proof in [CLPR]. While in that paper the authors are dealing with complex numbers and the well-known properties of the absolute value of a complex number we are working in the field $K(i)$ using the properties of the norm $\mathrm{N}_{K(i)/K}$.

LEMMA 2.1: *Let $x \in H(K)$, $a, y \in \mathrm{U}^+(K)$ and assume $a = x^2 + y^2$. Then there are $u, v \in H(K)$ such that:*

(i) $a = u^2 + v^2$.
(ii) $u \in \mathrm{U}^+(K)$.
(iii) $\frac{v^2}{u^2} \leq \frac{1}{9}\frac{x^2}{y^2}$.
(iv) *If $x \in \mathrm{U}^+(K)$ also v can be chosen in $\mathrm{U}^+(K)$.*

Proof: If $x = 0$ there is nothing to show. So assume $x \neq 0$. The field $L = K(i)$ where $i = \sqrt{-1}$ is a Galois extension of K of degree 2 with $1, i$ as a basis. We define $z = x + iy$ and $w = \frac{1}{2}x - iy$. Then $\mathrm{N}_{L/K}(z) = a$ and $\mathrm{N}_{L/K}(w) = \frac{1}{4}x^2 + y^2$. Let further $t = iw^2z$. An easy computation yields $t = b + ic$ where $b = y\left(\frac{3}{4}x^2 + y^2\right)$ and $c = \frac{1}{4}x^3$. Now we define

$$u = \frac{b}{\mathrm{N}_{L/K}(w)} \qquad \text{and} \qquad v = \frac{c}{\mathrm{N}_{L/K}(w)}.$$

By Lemma 1.1(ii) we have $b, \mathrm{N}_{L/K}(w) \in \mathrm{U}^+(K)$ and hence also $u \in \mathrm{U}^+(K)$. If further $x \in \mathrm{U}^+(K)$ then also $v \in \mathrm{U}^+(K)$. Since

$$b^2 + c^2 = \mathrm{N}_{L/K}(t) = \mathrm{N}_{L/K}(iw^2z) = \mathrm{N}_{L/K}(w)^2 a$$

we obtain $a = u^2 + v^2$. Now it is only left to show that $\frac{v^2}{u^2} \leq \frac{1}{9}\frac{x^2}{y^2}$. We have

$$b = y\left(\frac{3}{4}x^2 + y^2\right) \geq \frac{3}{4}x^2 y.$$

Hence

$$\frac{v^2}{u^2} = \frac{c^2}{b^2} \leq \frac{1}{9}\frac{x^2}{y^2}$$

and Lemma 2.1 is shown. ∎

This lemma was the major step towards the main result of this paper. The following corollary is an easy consequence of Lemma 2.1.

COROLLARY 2.2: *Let $x \in H(K)$, $a, y \in \mathrm{U}^+(K)$, $\varepsilon \in \mathbb{Q}^+$ and assume $a = x^2 + y^2$. Then there are $u, v \in H(K)$ such that:*

(i) $a = u^2 + v^2$.

(ii) $u, u+v, u-v \in \mathrm{U}^+(K)$.

(iii) $v^2 < \varepsilon u^2$.

(iv) *If $x \in \mathrm{U}^+(K)$ also v can be chosen in $\mathrm{U}^+(K)$.*

Proof: W.l.o.g. we may assume $\varepsilon < 1$. Since $y \in H(K)^\times$ we have $\frac{x}{y} \in H(K)$ and hence $\frac{x^2}{y^2} \le n$ for some natural number n. Successive application of Lemma 2.1 gives us $u, v \in H(K)$ such that $a = u^2 + v^2, u \in \mathrm{U}^+(K)$ and $v^2 < \varepsilon u^2$. Furthermore v can be chosen as a totally positive unit of $H(K)$ if x was one. Now we have to show that also $u \pm v \in \mathrm{U}^+(K)$. Since

$$(u+v)\cdot(u-v) = u^2 - v^2 > u^2 - \varepsilon u^2 = (1-\varepsilon)u^2 \in \mathrm{U}^+(K)$$

also $(u+v)\cdot(u-v) \in \mathrm{U}^+(K)$ by Lemma 1.1(ii). Thus $u \pm v \in H(K)^\times$. To show that both elements are sums of squares it suffices to show that they are positive w.r.t. every ordering of the field K.

Let P be an arbitrary ordering of K. Then $v \in P$ or $-v \in P$ and since u is a sum of squares we have $u+v \in P$ or $u-v \in P$. But

$$(u+v)\cdot(u-v) = u^2 - v^2 \in \mathrm{U}^+(K) \subset P \setminus \{0\}$$

and since at least one factor is positive also the other factor has to be positive w.r.t. P. Hence $u \pm v \in P$ and we are done. ∎

PROPOSITION 2.3: *Let $x_1, \ldots, x_n \in H(K), a, y \in \mathrm{U}^+(K), \varepsilon \in \mathbb{Q}^+$ and assume $a = \sum_{i=1}^{n} {x_i}^2 + y^2$. Then there exist $u_1, \ldots, u_{n+1} \in \mathrm{U}^+(K)$ such that*

(i) $a = \sum_{i=1}^{n+1} u_i^2$.

(ii) $u_i^2 < \varepsilon a \qquad (1 \le i \le n)$.

Proof: The proof is by induction on n. In the case $n = 0$ there is nothing to show.

$n = 1$: First we apply Corollary 2.2 and obtain elements $u_1, v_1 \in H(K)$ such that $a = u_1^2 + v_1^2$ and $u_1, u_1 \pm v_1 \in \mathrm{U}^+(K)$. Then $(u_1+v_1)^2 + (u_1-v_1)^2 = 2a$ and thus $2a$ is a sum of two squares of totally positive units of $H(K)$. Now we apply

Corollary 2.2 again and get elements $u_2, v_2 \in H(K)$ such that $2a = u_2^2 + v_2^2$ and $u_2, u_2 \pm v_2 \in \mathrm{U}^+(K)$. Then

$$a = \left(\frac{u_2 + v_2}{2}\right)^2 + \left(\frac{u_2 - v_2}{2}\right)^2$$

and so also a is a sum of two squares of totally positive units of $H(K)$. Now a further application of Corollary 2.2 gives us the desired result.

$n-1 \Rightarrow n$: First we apply the case $n = 1$ to x_1, y and obtain $u_1, z \in \mathrm{U}^+(K)$ such that $x_1^2 + y^2 = u_1^2 + z^2$ and $u_1^2 < \varepsilon(u_1^2 + z^2) \leq \varepsilon a$. Now let

$$b = \sum_{i=2}^{n} x_i^2 + z^2.$$

Then $a = b + u_1^2$. We can apply the induction hypothesis on b and obtain elements $u_2, \ldots, u_{n+1} \in \mathrm{U}^+(K)$ such that

$$b = \sum_{i=2}^{n+1} u_i^2 \qquad \text{and} \qquad u_i^2 < \varepsilon b < \varepsilon a \quad (2 \leq i \leq n+1).$$

and Proposition 2.3 is proved. ∎

Before we state and prove the main result of this section we need a further definition. The second Pythagoras number $\mathrm{P}_2(K)$ of a field K is the least natural number n such that every sum of squares of elements of K is already a sum of n squares. If such a number does not exist we let $\mathrm{P}_2(K) = \infty$.

THEOREM 2.4: *Let $a \in \mathrm{U}^+(K)$. Then there exists a natural number $n \leq \mathrm{P}_2(K)$ and elements $u_1, \ldots, u_{n+1} \in \mathrm{U}^+(K)$ such that*

$$a = \sum_{i=1}^{n+1} u_i^2.$$

Proof: As $a \in \mathrm{U}^+(K)$, also $a^{-1} \in \mathrm{U}^+(K)$, so $a^{-1} \leq l$ for some natural number l. W.l.o.g., $l = m^2$ for some $m \in \mathbb{N}$. Then there exist $x_1, \ldots, x_n \in K$ such that

$$a = \sum_{i=1}^{n} x_i^2 + \left(\frac{1}{m}\right)^2$$

We may assume $n \leq \mathrm{P}_2(K)$. From Lemma 1.1(i) we get $x_1, \ldots, x_n \in H(K)$. The assertion now follows from Proposition 2.3. ∎

3. Representation as sums of higher powers

In this section we prove that every totally positive unit of the real holomorphy ring of a formally real field is also a sum of $2n$-th powers of totally positive units for all natural numbers n. Our proof is based on certain polynomial identities, discovered by Hilbert. He used these so-called Hilbertian identities in his solution of Waring's Problem. We first have to prove the following lemma.

LEMMA 3.1: *Let n be a natural number. Then there exists a positive rational number ε such that for all sums $u \geq 1$ of $2n$-th powers of totally positive units of $H(K)$ and all $z \in \mathrm{U}^+(K)$ which satisfy $z < \varepsilon$ we have*

$$u + z^2 = \sum_{i=1}^{m} z_i^{2n}$$

for certain $m \in \mathbb{N}, z_1, \ldots, z_m \in \mathrm{U}^+(K)$

Proof: There exists a finite set $S \subset \mathbb{Q}^\times$ such that for all $k \in \mathbb{N}$ a polynomial identity of the form

$$\left(\sum_{i=0}^{k} X_i^2\right)^{n+1} = \sum_{j=1}^{s} a_j \left(\sum_{i=0}^{k} \alpha_{ij} X_i\right)^{2n+2} \tag{1}$$

for certain $s \in \mathbb{N}$, $a_1, \ldots, a_s \in \mathbb{Q}^\times, \alpha_{ij} \in S \quad (1 \leq i \leq k, 1 \leq j \leq s)$ holds. See e.g. [H]. We actually will make use of the fact that the set S only depends on n but not on k. Of course the numbers $s, a_1, \ldots, a_s$ depend on k. We choose a positive rational number c such that

$$c > \frac{|\alpha|}{|\beta|} \qquad \text{for all } \alpha, \beta \in S.$$

and let $\varepsilon \in \mathbb{Q}^+$ such that $\left(16c^2 + 2\right)\varepsilon^2 < 1$. This yields

$$16c^2\varepsilon^2 < 1 - 2\varepsilon^2 \tag{2}$$

which will be needed later. Since $0 < z^2 < \varepsilon^2 < \frac{1}{2}$ and $u \geq 1$ we have $u - z^2 \in \mathrm{U}^+(K)$. So we can apply Theorem 2.4 and obtain elements $x_1, \ldots, x_k \in \mathrm{U}^+(K)$ such that

$$u - z^2 = \sum_{i=1}^{k} x_i^2.$$

Since $u \in H(K)$ there exists a natural number t such that $t \pm u \in \sum K^2$ and hence $u \le t$. Now we apply Proposition 2.3 and get elements $y_1, \dots, y_k \in \mathrm{U}^+(K)$ such that

$$u - z^2 = \sum_{i=1}^{k} y_i^2 \qquad \text{and} \qquad y_i^2 < \left(\frac{\varepsilon}{tk}\right)^2 (u - z^2) < \left(\frac{\varepsilon}{k}\right)^2 \quad (2 \le i \le k).$$

So we have

$$y_i < \frac{\varepsilon}{k} \quad (2 \le i \le k) \qquad \text{and hence} \qquad \sum_{i=2}^{k} y_i < \varepsilon.$$

Further

$$1 - \varepsilon^2 < u - z^2 \le y_1^2 + (k-1)\left(\frac{\varepsilon}{k}\right)^2 \le y_1^2 + \varepsilon^2.$$

This implies $1 - 2\varepsilon^2 < y_1^2$ and together with (2) we obtain $4c\varepsilon < y_1$. Now look at the Hilbertian identity (1) and take twice the partial derivative w.r.t. X_0. We obtain the identity

$$\text{(3)} \quad \left(\sum_{i=0}^{k} X_i^2\right)^n + 2nX_0^2 \left(\sum_{i=0}^{k} X_i^2\right)^{n-1} = (2n+1) \sum_{j=1}^{s} a_j \alpha_{0j}^2 \left(\sum_{i=0}^{k} \alpha_{ij} X_i\right)^{2n}.$$

Let

$$w_j = \frac{\alpha_{0j}}{\alpha_{1j}} z + y_1 + \sum_{i=2}^{k} \frac{\alpha_{ij}}{\alpha_{1j}} y_i \quad (1 \le j \le s).$$

Then the substitution $X_0 \mapsto z, X_i \mapsto y_i \quad (1 \le i \le k)$ yields

$$u^n + 2nz^2 u^{n-1} = (2n+1) \sum_{j=1}^{s} a_j \alpha_{0j}^2 \alpha_{1j}^{2n} w_j^{2n}$$

and therefore

$$u + z^2 = \frac{2n+1}{2n} \frac{1}{u^{n-1}} \sum_{j=1}^{s} a_j \alpha_{0j}^2 \alpha_{1j}^{2n} w_j^{2n} + \frac{2n-1}{2n} u.$$

Since u is a sum of $2n$-th powers of totally positive units of $H(K)$ and all coefficients are positive rational numbers, it suffices to show that all w_j are totally positive units of $H(K)$. This will be a consequence of the following estimate:

$$w_j = \frac{\alpha_{0j}}{\alpha_{1j}} z + y_1 + \sum_{i=2}^{k} \frac{\alpha_{ij}}{\alpha_{1j}} y_i \ge y_1 - c\left(z + \sum_{i=2}^{k} y_j\right).$$

Since $0 < z < \varepsilon$ we get

$$w_j \geq y_1 - 2c\varepsilon = \frac{1}{2}y_1 + \frac{1}{2}(y_1 - 4c\varepsilon) \geq \frac{1}{2}y_1.$$

Now Lemma 1.1(ii) implies that $w_j \in \mathrm{U}^+(K) \quad (1 \leq j \leq s)$. This completes the proof of Lemma 3.1. ∎

COROLLARY 3.2: *Let $n \in \mathbb{N}$ and $a \in \mathrm{U}^+(K)$. Then $1 + a$ is a sum of $2n$-th powers of totally positive units of $H(K)$.*

Proof: First we apply Theorem 2.4 and obtain a representation

$$a = \sum_{i=1}^{r} u_i^2$$

for certain $r \in \mathbb{N}, u_1, \ldots, u_r \in \mathrm{U}^+(K)$. Since $a \in \mathrm{U}^+(K)$ there is a natural number m such that $a \leq m$. Now let ε be as in Lemma 3.1 and choose a natural number k such that $m/k^2 < \varepsilon$. Then we have

$$a = \sum_{i=1}^{r} u_i^2 = k^2 \sum_{i=1}^{r} \left(\frac{u_i}{k}\right)^2 = \sum_{i=1}^{s} v_i^2$$

for certain $s \in \mathbb{N}, v_1, \ldots, v_s \in \mathrm{U}^+(K)$ with $v_i^2 < \varepsilon \quad (1 \leq i \leq s)$. We now show by induction on t that

$$1 + \sum_{i=1}^{t} v_i^2$$

is a sum of $2n$-th powers of totally positive units of $H(K)$. The case $t = 0$ is clear. For the induction step apply Lemma 3.1. ∎

Now we are able to prove the main result of this paper.

THEOREM 3.3: *Every totally positive unit of $H(K)$ is a sum of $2n$-th powers of totally positive units of $H(K)$.*

Proof: Let $a \in \mathrm{U}^+(K)$. Then also $a^{-1} \in H(K)$ and hence $a^{-1} \leq m$ for some natural number m. Then $q = (m+1)a - 1 \geq a$. By Lemma 1.1(ii) we have $q \in \mathrm{U}^+(K)$. By Corollary 3.2, $(m+1)a = 1 + q$ is a sum of $2n$-th powers of totally positive units of $H(K)$. But then a is clearly also a sum of $2n$-th powers of totally positive units. ∎

4. Applications and Remarks

The first application of Theorem 3.3 is a positive answer to Schülting's question.

COROLLARY 4.1: *Let $n \in \mathbb{N}$ and $f, g \in \mathbb{R}[X]$ be of the same degree and without real zeroes. Assume that f/g is positive definite. Then there are $m \in \mathbb{N}$ and $f_i, g_i \in \mathbb{R}[X]$ $(1 \le i \le m)$ without real zeroes with $\deg f_i = \deg g_i$ $(1 \le i \le m)$ such that*

$$\frac{f}{g} = \sum_{i=1}^{m} \left(\frac{f_i}{g_i}\right)^{2n}.$$

Moreover if $n = 1$ we can take $m = 3$.

Proof: As

$$H(\mathbb{R}(X)) = \{h \in \mathbb{R}(X) |\ n \pm h \in \sum \mathbb{R}(X)^2 \text{ for some } n \in \mathbb{N}\}$$

one can easily check that a rational function h is an element of $H(\mathbb{R}(X))$ iff it is bounded on $\mathbb{R}$. Taking into account that the positive definite rational functions are exactly the sums of squares of elements of $\mathbb{R}(X)$, we find that a rational function is an element of $H(\mathbb{R}(X))^\times \cap \sum \mathbb{R}(X)^2$ iff it is positive definite and of the form f/g for some $f, g \in \mathbb{R}[X]$ without real zeroes and such that $\deg f = \deg g$. The first assertion now follows by Theorem 3.3. Since $\mathrm{P}_2(\mathbb{R}(X)) = 2$ (see e.g. Chap. 9, Cor. 2.4 and Chap. 11, Cor. 1.10 [L1]) we obtain $m = 3$ in the case $n = 1$ by Theorem 2.4. ∎

In the same way one defines the second Pythagoras number of a field one can define higher Pythagoras numbers of a field. The fourth Pythagoras number $\mathrm{P}_4(K)$ of the field K is the least natural number n such that every sum of fourth powers of elements of K is already a sum of n fourth powers. If such a number does not exist we let $\mathrm{P}_4(K) = \infty$. For the so-called pythagorean fields (i.e. fields with $\mathrm{P}_2(K) = 1$) it is shown in [S] that always $\mathrm{P}_4(K) \le 3$. The next corollary gives a better bound.

COROLLARY 4.2: *Let K be a formally real pythagorean field. Then $\mathrm{P}_4(K) \le 2$.*

Proof: Let $a \in K$ be a sum of fourth powers of elements of K. We have to show that a is a sum of two fourth powers. As $\mathrm{P}_2(K) = 1$, Prop. 2.11 [B2] yields $\varepsilon \in H(K)^\times$ and $b \in K$ such that $a = \varepsilon b^4$. Then also ε is a sum of fourth powers and hence a sum of squares, so $\varepsilon \in \mathrm{U}^+(K)$. Now we apply Theorem 2.4 and obtain $u_1, u_2 \in \mathrm{U}^+(K)$ such that $\varepsilon = u_1^2 + u_2^2$. Since $\mathrm{P}_2(K) = 1$, the elements u_1, u_2 are already squares. Hence $u_1 = x_1^2$ and $u_2 = x_2^2$ for certain $x_1, x_2 \in K$. Altogether this implies $a = (x_1 b)^4 + (x_2 b)^4$, and we are done. ∎

Prof. A. Prestel (Konstanz) pointed out an alternative proof of Theorem 2.4. His proof gives another bound for the number of summands in the representation and it only works for fields with finite second Pythagoras number but it is shorter and more conceptual than our proof. We will give a sketch of his proof:

Let M_K be the set of all real valued places of K. Then every $a \in H(K)$ acts by evaluation as a real valued function $\hat{a}$ on M_K:

$$\hat{a}\colon M_K \to \mathbb{R}$$
$$\lambda \mapsto \lambda(a)$$

M_K is quasi-compact w.r.t. the coarsest topology such that all functions $\hat{a}$ are continuous (Theorem 2.17 [B1]). Clearly

$$\Phi\colon H(K) \to C(M_K, \mathbb{R})$$
$$a \mapsto \hat{a}$$

is a homomorphism of rings. The image of Φ is dense in $C(M_K, \mathbb{R})$ w.r.t. the supremum norm(Theorem 2.20 [B1]). Further for $a \in H(K)$ we have:

$$\text{(1)} \qquad a \in \mathrm{U}^+(K) \Longleftrightarrow \hat{a}(\lambda) > 0 \quad \text{for all } \lambda \in M_K.$$

See 1.3 [B2]. Now let $a \in \mathrm{U}^+(K)$ and $n = \mathrm{P}_2(K)$. Choose some small positive rational number ε which will be fixed later. Let $f\colon M_K \to \mathbb{R}$ be a continuous function such that

$$\frac{1}{2n}(1-\varepsilon)\hat{a} < f^4 < \frac{1}{2n}\hat{a}.$$

Since $\mathrm{im}(\Phi)$ is dense in $C(M_K, \mathbb{R})$, there exists an element $b \in H(K)$ such that

$$\frac{1}{2n}(1-\varepsilon)\hat{a} < \hat{b}^4 < \frac{1}{2n}\hat{a}.$$

By (1) we have $b^4 \in \mathrm{U}^+(K)$. Now let $c = b^2$. Then also $c \in \mathrm{U}^+(K)$ and by (1)

$$\text{(2)} \qquad \frac{1}{2n}a - c^2,\ c^2 - \frac{1}{2n}(1-\varepsilon)a \in \mathrm{U}^+(K).$$

Hence there exist $x_1, \ldots, x_n \in H(K)$ such that

$$\frac{1}{2n}a = \sum_{i=1}^{n} x_i^2 + c^2.$$

Since $n = \mathrm{P}_2(K)$ it follows by multiplication by $2n$

$$a = 2\sum_{i=1}^{n} y_i^2 + 2nc^2 = 2\sum_{i=1}^{n} (y_i^2 + c^2) = \sum_{i=1}^{n} \left((c+y_i)^2 + (c-y_i)^2\right)$$

for certain $y_1, \ldots, y_n \in H(K)$. By (2),

$$\frac{1}{2n}a - c^2 \leq \frac{1}{2n}\varepsilon a$$

and hence

$$y_j^2 \leq \sum_{i=1}^{n} y_i^2 = n\sum_{i=1}^{n} x_i^2 = n\left(\frac{1}{2n}a - c^2\right) < \frac{1}{2}\varepsilon a.$$

If ε is sufficiently small we get $c \pm y_i \in \mathrm{U}^+(K)$ as in the proof of Corollary 2.2.

In the proof of Corollary 4.1. we used that a rational function $h \in \mathbb{R}(X)$ is an element of $H(\mathbb{R}(X))$ iff it is bounded on $\mathbb{R}$. One can easily see that this is also true for a real closed field R with an archimedean ordering instead of the field $\mathbb{R}$. Hence Corollary 4.1 holds not only for the field $\mathbb{R}$ of real numbers but also for any real closed field with an archimedean ordering. However, it is not clear whether Corollary 4.1 holds for an arbitrary real closed field R or not. To solve this problem one perhaps has to study the totally positive units of the relative real holomorphy ring $H_R(R(X))$,i.e., the intersection of all residually real valuation rings of $R(X)$ which contain the field R.

It is also not clear whether the bound of $\mathrm{P}_2(K) + 1$ summands in Theorem 2.4 is the best possible.

However, for pythagorean fields it is the best possible bound as the following example shows:

The field $\mathbb{Q}(\sqrt{2})$ has exactly two orderings induced from the two different embeddings of this field into $\mathbb{R}$. Let R_1, R_2 be real closures of $\mathbb{Q}(\sqrt{2})$ w.r.t. these two orderings in some fixed algebraic closure of $\mathbb{Q}$. Let $K = R_1 \cap R_2$. As an intersection of real closed fields, K is pythagorean, so $\mathrm{P}_2(K) = 1$. Clearly $2 \in \mathrm{U}^+(K)$. But 2 is not a square of a totally positive unit of $H(K)$. Otherwise 2 would be a fourth power and hence $\sqrt{2}$ or $-\sqrt{2}$ is a square, which is impossible by construction.

In the above example we have $\mathrm{P}_2(K) = 1$ and $\mathrm{P}_4(K) \neq 1$. So the bound in Corollary 4.2 is the best possible.

References

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